THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH4030 Differential Geometry
10 October, 2024 Tutorial

1. (Exercise 2-5.14 of [Car16]) The *gradient* of a differentiable function $f : S \to \mathbb{R}$ is a differentiable map $\mathrm{grad}(f) : S \to \mathbb{R}^3$ which assigns to each point $p \in S$ a vector $\mathrm{grad}(f)_p \in T_pS$ so that for all $v \in T_pS$,

Hint: write $\mathrm{grad}\, f_p = \alpha X_1 + \beta X_2$ for some $\alpha, \beta \in \mathbb{R}$. Then substitute

$$\mathrm{grad}(f)_p \cdot v = df_p(v).$$

$v = X_1, X_2$ here to set up a system of equations to solve for α, β.

   (a) Express $\mathrm{grad}(f)$ in terms of the coefficients of the first fundamental form and the partial derivatives of f on the local parameterisation $X : U \to S$ at $p \in X(U)$.

   Hint: apply Cauchy-Schwarz to $\mathrm{grad}(f)_p \cdot v = df_p(v)$

   (b) Let $p \in S$ and $\mathrm{grad}(f)_p \neq 0$. Show that $v \in T_pS$ with $|v| = 1$ satisfies

$$df_p(v) = \max\{df_p(u) : u \in T_p(S), |u| = 1\}$$

   if and only if $v = \dfrac{\mathrm{grad}(f)_p}{|\mathrm{grad}(f)_p|}$. (*Thus, $\mathrm{grad}(f)_p$ gives the direction of maximum variation of f at p.*)

2. (Exercise 2-5.11 of [Car16]) Let S be a surface of revoluion and C its generating curve. Let s be the arc length of C and denote $\rho(s)$ to be the distance to the rotation axis of the point of C corresponding to s.

   (a) (Pappus' Theorem) Show that the area of S is

$$2\pi \int_0^{\ell} \rho(s) ds$$

   where ℓ is the length of C.

   (b) Apply part (a) to compute the area of a torus of revolution.

3. (Exercise 2-6.1 of [Car16]) Let S be a regular surface covered by coordinate neighborhoods V_1 and V_2. Assume that $V_1 \cap V_2$ has two connected components, W_1, W_2, and that the Jacobian of the change of coordinates is positive in W_1 and negative in W_2. Prove that S is non-orientable.

1. (Exercise 2-5.14 of [Car16]) The *gradient* of a differentiable function $f : S \to \mathbb{R}$ is a differentiable map $\operatorname{grad}(f) : S \to \mathbb{R}^3$ which assigns to each point $p \in S$ a vector $\operatorname{grad}(f)_p \in T_pS$ so that for all $v \in T_pS$,

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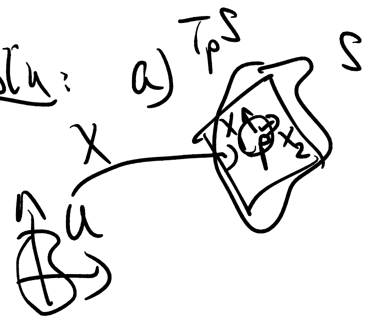
$$df_p(v) = \max\{df_p(u) : u \in T_p(S), |u| = 1\}$$

   if and only if $v = \dfrac{\operatorname{grad}(f)_p}{|\operatorname{grad}(f)_p|}$. (*Thus, grad*$(f)_p$ *gives the direction of maximum variation of* f *at* p.)

Sol'n: a)



We write $\operatorname{grad}(f)_p = \alpha X_1 + \beta X_2$ for some $\alpha, \beta \in \mathbb{R}$.

We have for $v = X_1$,

$$\operatorname{grad}(f)_p \cdot X_1 = df_p(X_1) = f_1 \quad \text{where } f_1 = \frac{\partial f}{\partial u_1}$$

$$\| \qquad\qquad$$

$$(\alpha X_1 + \beta X_2) \cdot X_1$$

$$\|$$

$$\alpha g_{11} + \beta g_{21}$$

So we have $\alpha g_{11} + \beta g_{21} = f_1$

Substituting $v = X_2$, get

$\alpha g_{12} + \beta g_{22} = f_2$.

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \quad (*)$$

Solving $(*)$, we get

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{1}{g_{11}g_{22} - g_{12}^2} \begin{pmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

Note: if we choose $g_{ij} = \delta_{ij}$ (Euclidean space)

i.e.

$$\operatorname{grad}(f)_p = \frac{f_1 g_{22} - f_2 g_{12}}{g_{11}g_{22} - g_{12}^2} X_1 + \frac{f_2 g_{11} - f_1 g_{12}}{g_{11}g_{22} - g_{12}^2} X_2$$

b) By Cauchy-Schwarz, we have

$$\begin{aligned} df_p(v) &= \operatorname{grad}(f)_p \cdot v \\ &\leq |\operatorname{grad}(f)_p \cdot v| \\ &\leq |\operatorname{grad}(f)_p| \cdot \cancel{|v|}^{1} = |\operatorname{grad}(f)_p| \quad \text{when } v \text{ is a unit vector.} \end{aligned}$$

So maximum is attained iff v and $\operatorname{grad}(f)_p$ are linearly dependent,

i.e. $v = \lambda \operatorname{grad}(f)_p$ for some $\lambda \in \mathbb{R}$.

v is a unit vector, so $\lambda = \dfrac{\pm 1}{|\operatorname{grad}(f)_p|}$

2. (Exercise 2-5.11 of [Car16]) Let S be a surface of revoluion and C its generating curve. Let s be the arc length of C and denote $\rho(s)$ to be the distance to the rotation axis of the point of C corresponding to s.

   (a) (Pappus' Theorem) Show that the area of S is
   $$2\pi \int_0^\ell \rho(s)ds$$
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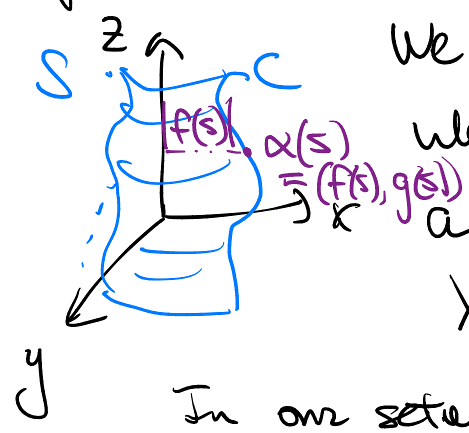
Sol'n: Wlog, can take C to be in the xz-plane and axis of rotation to be the z-axis.



We write curve C to be $\alpha(s) = (f(s), g(s))$ where s is arclength.

and S by
$$X(\theta, s) = (f(s)\cos\theta, f(s)\sin\theta, g(s))$$

In our setup, $\rho(s) = |f(s)|$.

$$X_\theta = (-f(s)\sin\theta, f(s)\cos\theta, 0)$$
$$X_s = (f'(s)\cos\theta, f'(s)\sin\theta, g'(s)).$$

So $g_{\theta s} = \begin{pmatrix} f^2(s) & 0 \\ 0 & \underbrace{f'(s)^2 + g'(s)^2}_{= 1} \end{pmatrix}$

by arclength $(|\alpha'(s)| \equiv 1)$.

So
$$A = \iint_{X^{-1}(S)} \sqrt{\det g_{\theta s}}\, d\theta\, ds = \int_0^\ell \int_0^{2\pi} \sqrt{f^2(s)}\, d\theta\, ds$$
$$= 2\pi \int_0^\ell |f(s)| ds = 2\pi \int_0^\ell \rho(s) ds.$$

b) For a torus of revolution rotating a circle centred at $(a, 0)$ with radius $r > 0$

$$A = 4\pi r^2 a.$$

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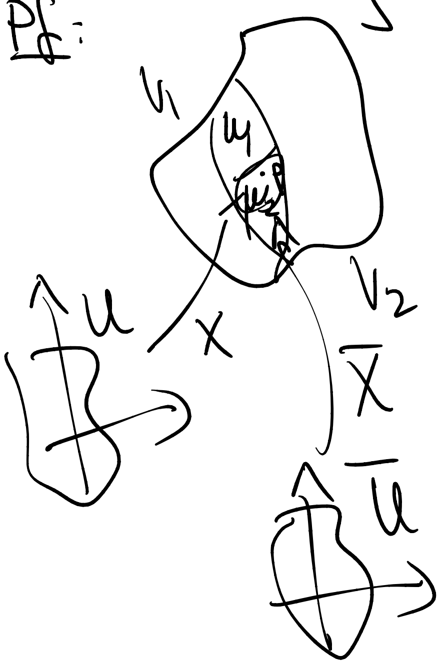
Pf:



Sps for the sake of contradiction that S is orientable. Then by lemma 3.17 of lecture notes, there is a smooth global choice of unit normal $N: S \to \mathbb{R}^3$.

$$N(p) = \frac{(X_u \times X_v)(p)}{|(X_u \times X_v)(p)|} \quad \text{for } p = X(u_0, v_0),\ (u_0, v_0) \in \mathcal{U}.$$

s.t. $X(\mathcal{U}) \subseteq V_1$.

OTOH, $$N(p) = \frac{(\overline{X}_{\bar{u}} \times \overline{X}_{\bar{v}})(p)}{|(\overline{X}_{\bar{u}} \times \overline{X}_{\bar{v}})(p)|} \quad \text{for } p = \overline{X}(\bar{u}_0, \bar{v}_0),$$ $(\bar{u}_0, \bar{v}_0) \in \overline{\mathcal{U}}$ and $\overline{X}(\overline{\mathcal{U}}) \subseteq V_2$.

But for $p \in W_2$, we have

$$N(p) = \frac{(\overline{X}_{\bar{u}} \times \overline{X}_{\bar{v}})(p)}{|(\overline{X}_{\bar{u}} \times \overline{X}_{\bar{v}})(p)|} = \frac{(X_u \times X_v)(p)}{|(X_u \times X_v)(p)|} \underbrace{\operatorname{sign}\left(\frac{\partial(u,v)}{\partial(\bar{u},\bar{v})}\right)}_{=-1}$$

$$= -\frac{(X_u \times X_v)(p)}{|(X_u \times X_v)(p)|} = -N(p).$$

a contradiction.